TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 235, January 1978

REAL STRUCTURE IN COMPLEX L_1-PREDUALS([1])

BY

DANIEL E. WULBERT

ABSTRACT. Call a complex Banach space selfadjoint if it is isometrically isomorphic to a selfadjoint subspace of a $C(X, \mathbf{C})$-space. B. Hirsberg and A. Lazar proved that if the unit ball of a complex Lindenstrauss space, E, has an extreme point, then E is selfadjoint. Here we will give a characterization of selfadjoint Lindenstrauss spaces, and construct a nonselfadjoint complex Lindenstrauss space.

1. **Introduction.** If $E = C(X, \mathbf{R})$ (or a sublattice of $C(X, \mathbf{R})$) then it is a result of the classical Kakutani characterization of L lattices [12] that there is a measure space (X, Σ, μ) such that E^* is isometrically isomorphic with $L_1(X, \Sigma, \mu, \mathbf{R})$. In 1955 A. Grothendieck [9] initiated the formal study of the class of Banach spaces which have L_1-duals, and conjectured a functional representation for such spaces. In 1964 J. Lindenstrauss [20] showed that these L_1-preduals were, in a sense, the solution to the problem of the extendibility of compact operators. Lindenstrauss' development involved a detailed study of the structure of L_1-preduals, and, for example, settled Grothendieck's conjecture as well as several others. Since then, extensive investigations of this class of spaces have been done in papers such as [2], [7], [8], [15]–[18], [21], [24], [27] and [30] (for extensive bibliographies see [14] and [19]). In fact, now an argument is frequently made, that–along with the L_p-spaces $(1 \leq p \leq \infty)$–the L_1-predual spaces should be considered classical Banach spaces. These Banach spaces–whose duals are L_1-spaces–are now also called *Lindenstrauss spaces*.

In this article we are interested in relating a complex Lindenstrauss space E with the existence and structure of its real sections: the real Lindenstrauss subspaces of E whose complex spans are dense in E. This is related to the work of B. Hirsberg and A. Lazar who proved the following:

1.1. THEOREM. *Let E be a subspace of $C(X, \mathbf{C})$ which contains the constants. The following are equivalent*:

(a) *E is a Lindenstrauss space*,

Received by the editors March 17, 1976.
AMS (MOS) subject classifications (1970). Primary 46-10; Secondary 46-25, 46-35, 46-55.
Key words and phrases. Lindenstrauss space, Hirsberg-Lazar theorem.
([1])This research was supported by the National Science Foundation.

(b) *E is selfadjoint and* Re *E is a real Lindenstrauss space.*

1.2. THEOREM. *If E is a complex Lindenstrauss space, and $u \in \operatorname{ext} S(E^*)$ then there is a compact set X and an isometric isomorphism T of E into $C(X, \mathbf{C})$ such that $Tu = 1$.*

A main result of this paper is an example of a complex Lindenstrauss space which is not isometric with a selfadjoint subspace of some $C(X, \mathbf{C})$ (Example 6.1). The proof depends on a characterization of those complex Lindenstrauss spaces which are isometric to selfadjoint function spaces (Theorem 5.3). Among the consequences of the development is a short proof of Theorem 1.1.

II. **Preliminaries and notation.** We will single out three theorems on the structure of Lindenstrauss spaces that will be used in the development of this paper. All three results apply to either real or complex Banach spaces.

2.1. THEOREM. *If S is a countable subset of a Lindenstrauss space E, then there is a separable Lindenstrauss space F such that $S \subseteq F \subseteq E$.*

2.2. THEOREM. *A separable Banach space E is a Lindenstrauss space if and only if there are subspaces $E_1 \subseteq E_2 \subseteq \ldots$ such that*
(1) *E_n is isometrically isomorphic to l_∞^n, and*
(2) *$\bigcup E_n$ is dense in E.*

2.3. THEOREM. *E is a Lindenstrauss space if and only if for each $\varepsilon > 0$ and each finite set $S \subseteq E$ there is a finite dimensional subspace $F \subseteq E$ such that*
(1) *F is isometrically isomorphic with l_∞^n for some n, and*
(2) $\sup\{\inf\{\|s - f\| : f \in F\} : s \in S\} < \varepsilon$.

Theorem 2.1 was first discovered by Hirsberg and Lazar who observed that this fact could be gleaned from a proof in [E. Michael and A. Pełczynski, 1966]. More recently E. Lacey isolated a short proof [Lacey, 1974, p. 232, Lemma 1].

Theorem 2.2 was proved for a special case by E. Michael and A. Pełczynski [1966], and completed by A. Lazar and J. Lindenstrauss [1966]. Theorem 2.3 is a minor variant of results in the literature. The sufficiency assumes more than the sufficient condition established by Lazar and Lindenstrauss [1966, Theorem 1, (iii) $\Rightarrow$ (i)]. The necessity follows from Theorems 2.1 and 2.2.

We will use the rest of the section to record our notation. We will use $\mathbf{R}$ and $\mathbf{C}$ to denote the real and complex fields. For a compact Hausdorff space X, $C(X, \mathbf{C})$ represents the continuous complex valued functions on X, equipped with the supremum norm, and $l_\infty^n(\mathbf{C})$ is equivalent to $C(Y, \mathbf{C})$ for a set Y containing n points. For a measure space (X, Σ, μ), $L_1(X, \Sigma, \mu, \mathbf{C})$ consists of the integrable complex valued functions on X with the usual L_1 norm. We will often write $L_1(\mu, \mathbf{C})$ for $L_1(X, \varepsilon, \mu, \mathbf{C})$. Analogous definitions hold for the real Banach spaces obtained when replacing **C** by **R** above.

For a Banach space E, E^* and $S(E)$ represent the dual of E and the unit ball of E respectively. The extreme points of a convex set K is the set ext(K). If f is a function on a set X and $x \in X$, $\delta(x)f = f(x)$, and $\delta(x)$ is called the point evaluation at x. The characteristic function of a set $A \subseteq X$ is

$$\psi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$$

and the Dirac delta function is

$$\delta_{ij} = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

Let E be a Banach space $M \subseteq E$, and $f \in E$; then $d(f, M) = \inf\{\|f - m\|: m \in M\}$.

III. **Lindenstrauss subspaces of** $C(X, \mathbf{C})$. In this section we give a short proof of an extended version of Theorem 1.1 by Hirsberg and Lazar. Both proofs depend on specialized results from the literature. The argument here uses Theorems 2.1, 2.2, and 2.3 from the last section and circumvents the additional use of simplex structure employed in [Hirsberg and Lazar, 1973].

3.1. PROPOSITION. *Let A be a subspace of $C(X, \mathbf{C})$ that contains the constants. If A is an L_1-predual then A is selfadjoint.*

PROOF. Let f be in A, and let B be a separable L_1-predual such that $\{1, f\} \subseteq B \subseteq A$ [Theorem 2.1]. We will show $\bar{f} \in A$ by showing that B is selfadjoint. Let Q be the quotient space of X associated with the equivalence on X defined by $x = y$ if $b(x) = b(y)$ for all $b \in B$. Viewing B as a subspace of $C(Q, \mathbf{C})$ it suffices to show B is selfadjoint. Let

$$D = \text{closure}\left[\text{span Re } B\right] \subseteq C(Q, \mathbf{C}). \tag{3.1}$$

Then

$$\partial B = \partial \text{ Re } B = \partial \text{Re } D = \partial D, \tag{3.2}$$

where ∂ denotes the choquet boundary of a space.

Since B is a separable Lindenstrauss space, there is a sequence of norm one projections P_i of D into B such that $P_i b \to b$ for all $b \in B$ [Theorem 2.2]. This implies that $(P_i d)(x) \to d(x)$ for $x \in \partial B$ and $d \in D$ [D. Wulbert, 1966, Lemma 1]. From choquet theory and the Lebesgue dominated convergence theorem, this in turn implies $P_i d \to d$ weakly (since $\partial B = \partial D$) [R. R. Phelps, 1966]. Since $P_i d \in B$, we conclude $D \subseteq B$, and B is selfadjoint. □

3.2. LEMMA. *Let G be a real linear subspace of $C(X, \mathbf{R})$ that is isometric to $l_\infty^n(\mathbf{R})$. Then the complex span of G is isometric to $l_\infty^n(\mathbf{C})$.*

3.3. LEMMA. *Let G be a complex linear subspace of $C(X, \mathbf{C})$ that is linearly* *isometric to* $l_\infty^n(\mathbf{C})$. *There is a subset* $S \subseteq \operatorname{Re} G$ *such that*

(1) S *is real linearly isometric to* $l_\infty^n(\mathbf{R})$, *and*

(2) *if* f *is a real function such that* $d(f, G) < \varepsilon$ *then* $d(f, S) < 2\varepsilon$.

Later we will establish an extension of Lemma 3.2. However this case can be done simultaneously with the proof of Lemma 3.3, and is the case needed for the Hirsberg-Lazar theorem.

Proof. We will prove both lemmas with the same argument. For either scalar field, there is a basis $\{g_i\}^n \subseteq G$ and extreme linear functionals $\{k_i\}^n$ of the unit ball of G^* such that:

(1) $k_i g_j = \delta_{i,j}$, and

(2) for all functionals k with $\|k\| \leqslant 1$ there are scalars λ_i such that both $\Sigma|\lambda_i| \leqslant 1$ and $kg = \Sigma \lambda_i k_i g$ for $g \in G$.

Since k_i is an extreme functional there is an $x_i \in X$ and a scalar s_i such that $|s_i| = 1$ and $k_i g = s_i g(x_i)$ for $g \in G$. Replacing g_i by $s_i^{-1} g_i$ there is a basis–which we assume is already $\{g_i\}$–such that:

(a) $g_i(x_j) = \delta_{ij}$, and

(b) for each $y \in X$ there are scalars λ_i such that both

(i) $\Sigma|\lambda_i| \leqslant 1$, and (ii) $\Sigma \alpha_i g_i(y) = \Sigma \lambda_i \alpha_i$ for any set of scalars α_i.

Lemma 3.2 follows easily by applying (b) to both the real and imaginary parts of functions in the complex span of G.

To prove Lemma 3.3 let

$$S = \text{real span } \{\operatorname{Re} g_j\}_{j=1}^n. \tag{3.3}$$

It follows that the map

$$f \to (f(x_1), f(x_2), \ldots, f(x_n)) \tag{3.4}$$

is a linear isometry of S onto $l_\infty^n(\mathbf{R})$. For if $x \in X$ and $\alpha_j \in \mathbf{R}$

$$\left|\sum \alpha_j \operatorname{Re} g_j(x)\right| \leqslant \left|\sum \alpha_j g_j(x)\right| \leqslant \left|\sum \lambda_j \alpha_j\right| \leqslant \max|\alpha_j|. \tag{3.5}$$

Finally to verify the estimate suppose f is a real function and $\alpha_j \in \mathbf{C}$ are such that $\|f - \Sigma \alpha_j g_j\| \leqslant \varepsilon$. Thus

$$|\operatorname{Im} \alpha_j| \leqslant \left\|\left(f - \sum \alpha_m g_m\right)(x_j)\right\| \leqslant \varepsilon, \tag{3.6}$$

and since each function in G attains its maximum at some x_j,

$$\left\|\sum \alpha_j g_j - \sum (\operatorname{Re} \alpha_j) g_j\right\| \leqslant \varepsilon. \tag{3.7}$$

Thus if

$$q = \operatorname{Re} \sum (\operatorname{Re} \alpha_j) g_j, \tag{3.8}$$

we have that

$$\left\|\sum \operatorname{Re}(\alpha_j g_j) - q\right\| < \varepsilon, \tag{3.9}$$

but $\|\Sigma \mathrm{Re}(\alpha_j g_j) - f\| < \varepsilon$. Hence $\|q - f\| \leq 2\varepsilon$. □

3.4. PROPOSITION. *Let A be a selfadjoint subspace of $C(X, \mathbf{C})$. Then A is a complex Lindenstrauss space if and only if* Re *A is a real Lindenstrauss space.*

PROOF. This is now immediate from the lemmas and Theorem 2.3 in the previous section.

In fact we also have some Lemma 3.3 and Theorem 2.3 that:

3.5. PROPOSITION. *If $A \subseteq C(X, \mathbf{C})$ is a complex Lindenstrauss space then* Re *A is a real Lindenstrauss space.*

3.6. COROLLARY. *Suppose Y and Z are disjoint open subsets of X whose union is X. If A is a real Lindenstrauss subspace of $C(X, \mathbf{R})$ then*

$$E = \{f \in C(X, \mathbf{R}): f|_Y \in A|_Y \text{ and } f|_Z \in A|_Z\}$$

is also a real Lindenstrauss space.

PROOF. Let $g = 1$ on Y and i on Z.

Now by Proposition 3.4, $F =$ complex span $[A]$ is a complex Lindenstrauss space. Hence, so is gF. By Proposition 3.5 then, $E = \mathrm{Re}(gF)$ is a real Lindenstrauss space. □

3.7. COROLLARY (HIRSBERG-LAZAR). *Let A be a subspace of $C(X, \mathbf{C})$ which contains the constants. The following are equivalent*:

(1) *A is a complex Lindenstrauss group*;

(2) *A is selfadjoint, and* Re *A is a real Lindenstrauss space.*

3.8. COROLLARY. *Let A be a selfadjoint complex Lindenstrauss space contained in $C(X, \mathbf{C})$. The subspaces E_n and F of Theorems* 2.2 *and* 2.3 *respectively can be chosen to be selfadjoint.*

PROOF. The result is obtained by applying Proposition 3.4, the real versions of Theorems 2.2 and 2.3 and Lemma 3.2.

IV. **Real sections of a complex Lindenstrauss space.**

DEFINITION. A real linear subspace G of a complex Banach space E is *determined by real functionals* (or *real determined* for short) if there is a set of norm one functionals $M \subseteq E^*$ such that

(1) for $g \in$ complex span G, $|g| = \sup\{|g(m)|: m \in M\}$, and

(2) m has only real values on G, for each $m \in M$.

Comment. (1) Any subspace of real functions in $E = C(X, \mathbf{C})$ is real determined. Here the point evaluation functionals satisfy the role of M in the definition above.

(2) A real linear subspace of a real determined space is also real determined.

4.1. LEMMA. *Let G be a real determined subspace of a complex Banach space. A real linear isometry of G onto $l_\infty^n(\mathbf{R})$ can be extended to a complex linear isometry of the complex span of G onto $l_\infty^n(\mathbf{C})$.*

PROOF. Since G is isometric to $l_\infty^n(\mathbf{R})$ there is a basis $\{g_j\}_{j=1}^n \subseteq G$ and $\{k_j\}_{j=1}^n \subseteq \operatorname{ext} S(G^*)$ such that (1) $k_s(g_t) = \delta_{s,t}$, and (2) $S(G^*)$ is contained in the closed balanced convex hull of $\{k_j\}_{j=1}^n$.

The functionals k_j can be extended to norm one real linear functionals $\tilde{k}_j$ acting on the real linear space, $\tilde{G}$ = complex span G. Also $\tilde{k}_i$ is the real part of the complex linear functional

$$K_j(g) = \tilde{k}_j(g) - i\tilde{k}_j(ig). \tag{4.1}$$

Of course, K_j also has norm one since if $g \in \tilde{G}$ and $s \in \mathbf{C}$ is such that

$$|s| = 1 \quad \text{and} \quad sK_j(g) = |K_j(g)|, \tag{4.2}$$

we have

$$|K_j g| = K_j(sg) = \tilde{k}_j(sg) \leqslant \|g\|. \tag{4.3}$$

For any $\{\lambda_m\}_{k=1}^n \in \mathbf{R}$ such that $\sup|\lambda_m| \leqslant 1$, we have

$$1 \geqslant \left|K_j\left(\sum^n \lambda_m g_m\right)\right| \geqslant k_j\left(\sum \lambda_m g_m\right) = \lambda_j. \tag{4.4}$$

Taking $\lambda_m = \delta_{m,j}$ gives $K_j(g_j) = 1$. This, with the inequality, implies that $K_j(g_m) = \delta_{j,m}$. Hence the mapping

$$f \to (K_1 f, K_2 f, \ldots, K_n f) \tag{4.5}$$

is a norm decreasing mapping of $\tilde{G}$ onto $l_\infty^n(\mathbf{C})$. Furthermore, on G, it agrees with the original isometry onto $l_\infty^n(\mathbf{R})$.

Now suppose L is a norm one complex functional on $\tilde{G}$, and $g \in G$. The proof will be completed by showing that

$$|Lg| \leqslant \max|K_j(g)|. \tag{4.6}$$

Again let $s \in \mathbf{C}$ be such that $|s| = 1$, and $sLg = |Lg|$. Then by condition (2) there are $\lambda_i \in \mathbf{R}$ with $\sum|\lambda_i| \leqslant 1$ and

$$\begin{aligned} |Lg| &= sLg = [\operatorname{Re}(sL)](g) = \sum \lambda_j k_j(g) \\ &= \sum \lambda_j \tilde{k}_i(g) = \sum \lambda_j \operatorname{Re}[K_j(g)] \\ &= \sum |\lambda_j|\, |K_j(g)| \leqslant \max|K_j(g)|. \quad \square \end{aligned} \tag{4.7}$$

The condition that G is real determined, in the last lemma, cannot be simply dropped. Åsvald Lima has shown me the following clever example.

EXAMPLE (LIMA). Let $X = \{1, 2, 3, 4\}$, and let $E \subseteq C(X, \mathbf{C})$ be spanned by $f_1 = (1, 1, 1, 1)$, $f_2 = (1, 1, -1, -1)$ and $f_3 = (1, -1, 1, -1)$, i.e., $E = \{f:$

$f(1) + f(4) = f(2) + f(3)\}$. Then E is isometric with $\mathbf{C}^3$ equipped with the norm $\|(\alpha_1, \alpha_2, \alpha_2)\| = \max|\alpha_1 \pm \alpha_2 \pm \alpha_3|$ under the map $T(\alpha_1, \alpha_2, \alpha_3) = \sum_{i=1}\alpha_i f_i$.

Let $p_i f = f(i)$ be the point evaluation at i, and let

$$k_1 = p_1 + p_2 + p_3 + p_4, \quad k_2 = p_1 - p_2 + p_3 - p_4, \quad k_3 = p_1 + p_2 - p_3 - p_4.$$

Define the mapping H by $H(1, 0, 0) = k_1$, $H(0, 1, 0) = k_2$, and $H(0, 0, 1) = k_3$. The linear extension to $l_\infty^n(\mathbf{R})$ is then shown to be an isometric mapping onto the real span of k_1, k_2 and k_3.

However the complex span of k_1, k_2, and k_3 is E^* which is not isometric with $l_\infty^n(\mathbf{C})$ since E is not isometric with $l_1^3(\mathbf{C})$.

4.2. LEMMA. *Let G be a real determined subspace of a complex Banach space E. If G is isometric to a real Lindenstrauss space then the closed complex span of G is a complex Lindenstrauss space.*

PROOF. If $g_1, \dots, g_n$ is a finite set in the complex span of G, there are $h_1, \dots, h_m$ in G whose span contains $g_1, \dots, g_n$. Since $h_1, \dots, h_m$ can be approximated arbitrarily closely by an isometric copy of $l_\infty^k(\mathbf{R})$, for some k, the last lemma shows that $g_1, \dots, g_n$ can be approximated by $l_\infty^k(\mathbf{C})$. From Theorem 2.3 the closed complex span of G is a Lindenstrauss space. □

4.3. COROLLARY. *Let $A \subseteq C(X, \mathbf{C})$ be a function algebra. Suppose A contains a point separating real determined subspace B such that* (1) $1 \in B$ *and* (2) *B is a real L_1-predual Banach space. Then $A = C(X, \mathbf{C})$.*

PROOF. Lemma 4.2 shows that the complex space of B is a complex L_1-predual. So Proposition 3.1 gives us that A contains a point separating selfadjoint subspace. Therefore the antisymmetric subsets for A are singletons, and by Bishops' generalization of the Stone-Weierstrass theorem [E. Bishop, 1961], $A = C(X, \mathbf{C})$. □

DEFINITION. A subset G of a Banach space E is a *real section* if (1) G is a real Lindenstrauss space, (2) the complex span of G is dense in E, and (3) G is real determined in E.

Lemma 4.2 shows that if E has a real section then E is a complex Lindenstrauss space. We will show that a real section of a Lindenstrauss space plays the role of the real part of a selfadjoint subspace of a $C(X, \mathbf{C})$.

4.4. LEMMA. *Let G be a real section of a complex Lindenstrauss space E. For each f in E there is a unique decomposition $f = g + ih$ with g and h in G. Furthermore $\|g\| \leq \|f\|$.*

PROOF. We will first show that for $f \in E$ there are g and h in G such that $f = g + ih$. We already know–by definition–that $G + iG$ is dense in E. Let $G_n \subseteq G_{n+1} \subset \dots$ be real subspaces of G such that:

(1) each G_n is isometric to $l_\infty^m(\mathbf{R})$ for some m,

(2) dist(f, complex span G_n) $\to 0$ with n.

Hence there are g_n and h_n in G_n such that $g_n + ih_n \to f$. By Lemma 4.1, if g_m, h_m are in G_m then $\|g_m\| \leq \|g_m + ih_m\|$. Hence both $\{g_n\}$ and $\{h_n\}$ are cauchy sequences and converge to say g and h in G. It follows that $f = g + ih$. Also

$$\|g_n\| \leq \|g_n + ih_n\| \to \|f\|. \tag{4.8}$$

So $\|g\| \leq \|f\|$.

Finally we will observe that the representation is unique. Suppose $q_1 + iq_2 = q_3 + iq_4$ with each q_j in G. Let $\varepsilon > 0$ and let M be an isometric copy of $l_\infty^m(\mathbf{R})$ containing members m_j for which $\|m_j - q_j\| < \varepsilon$ for $j = 1, 2, 3$, and 4. Then

$$\|m_1 + im_2 - m_3 + im_4\| < 2\varepsilon \tag{4.9}$$

implying that $\|m_1 - m_2\|$ and $\|m_2 - m_4\|$ are less than 2ε and so $\|q_1 - q_3\|$ and $\|q_2 - q_4\|$ are less than 4ε. Since $\varepsilon > 0$ was arbitrary q_1 and q_3 must be equal to q_2 and q_4 respectively.

Hence we have shown that for each f in E there is a unique decomposition $f = g + ih$ with g and h in G, also $\|g\| \leq \|f\|$. The proof is completed. □

4.5. Lemma. *A real linear isometry between real sections of complex L_1-preduals extends to a complex linear isometry between the entire spaces.*

Proof. Let T be a real linear isometry of a real section V of a complex L_1-predual E onto a real section W of a complex predual F. Let $G \subseteq E$ be isometric to $l_\infty^n(\mathbf{R})$. Of course then $TG \subseteq W$ is also isometric to $l_\infty^n(\mathbf{R})$. For g and $h \in G$ let $T(g + ih) = Tg + iTh$. By Lemma 4.4 this mapping is well defined. By Lemma 4.1, T is a complex isometry of $G + iG$ onto $TG + iTG$. Since the unions of such spaces $G + iG$ and $TG + iTG$ are dense in E and F respectively, T extends to an isometry of E onto F. □

4.6. Example. A complex L_1-predual can have two nonisometric real sections. Let V be the subspace of continuous real functions on $[0, 2\pi]$ such that $f(x) = -f(x + \pi)$ for $x \in [0, \pi]$. Let $E \subseteq C([0, 2\pi], \mathbf{C})$ be the complex span of V. It is known that V is a real L_1-predual (in fact it is the range of a norm one projection on $C([0, 1], \mathbf{R})$). Hence by Lemma 4.2 E is a complex L_1-predual. Let T be the isometric mapping of $C([0, 2\pi], \mathbf{C})$ into itself defined by

$$(Tf)(x) = (\cos(x) + i\sin(x))f(x). \tag{4.10}$$

Then $F = T(E)$ is a complex L_1-predual containing 1. Hence it is selfadjoint. We conclude that $T^{-1}(\operatorname{Re} F)$ is a real section of E whose unit ball contains an extreme point. Since the unit ball of the real section V does not contain an extreme point, V and $T^{-1}(\operatorname{Re} F)$ cannot be isometric. □

In this example the space E admits an extreme point for its unit ball, although the real section V does not. Example 6.3 in §VI is relevant to this fact.

Although real sections of a Lindenstrauss space are not isometric, we will show later that they do have isometric dual spaces (Corollary 5.6). We do not know if two real sections are isomorphic to each other.

4.7. LEMMA. *Let E be a complex L_1-predual with a real section V. Then there is a compact Hausdorff space X, a selfadjoint L_1-predual $F \subseteq C(X, \mathbf{C})$, and a complex linear isometry of E onto F whose restriction to V is a real isometry onto the real functions in F.*

PROOF. Let X be the closure of $\operatorname{ext}(S(V^*))$ in the weak topology induced by V. In the usual way V embeds (real) isometrically onto a subspace of real functions in $C(X, \mathbf{C})$. Now Lemmas 4.2 and 4.5 complete the proof. □

4.8. COROLLARY. *Let A be a real determined real L_1-predual embedded in a complex Banach space E. If $f \in A$ then $(if) \notin A$.*

PROOF. We may assume that $E = A + iA$. Let T be the isometry of Lemma 4.7 mapping E into $C(X, \mathbf{C})$. Then if $f \in A$, TF is a real function, $T(if)$ is not, and so (if) cannot be in A. □

Let E be a complex Banach space. We will use $(E, \mathbf{R})$ to denote the real Banach space obtained by restricting the scalar field to $\mathbf{R}$. For $L \in E^*$ the mapping of L onto its real part is a real linear isometry of E^* onto $(E, \mathbf{R})^*$. Hence we observe the following:

4.9. LEMMA. $(E, \mathbf{R})^* = (E^*, \mathbf{R}) = \{\operatorname{Re} L \colon L \in E^*\}$.

4.10. LEMMA. *Let $E^* = L_1(\mu, \mathbf{C})$. If $L_1(\mu, \mathbf{R})$ is w^*-closed in $L_1(\mu, \mathbf{C})$ then*

$$F = \{f \in E \colon \operatorname{Re} f \perp L_1(\mu, \mathbf{R})\} \tag{4.11}$$

is a real section of E. Here $\operatorname{Re} f$ *denotes the real part of the complex linear functional on $L_1(\mu, \mathbf{C})$ associated with f.*

PROOF. From Lemma 4.9 we may identify $(E, \mathbf{R})$ with the real parts of a real linear subspace of E^{**}. Then F is the preannihilator of $L_1(\mu, \mathbf{R}) \subseteq (E, \mathbf{R})^*$. Since $L_1(\mu, \mathbf{R})$ is w^*-closed, $F^{\perp} = L_1(\mu, \mathbf{R})$, and

$$F^* = (L_1(\mu, \mathbf{C}), R)/L_1(\mu, \mathbf{R}) \tag{4.12}$$

but this is isometric with the purely imaginary functions in $L_1(\mu, \mathbf{C})$ (since $\operatorname{dist}[f, \operatorname{Re} L_1(\mu)] = \|\operatorname{Im} f\|$). Hence F is a real L_1-predual. To see that the complex span of F is dense in E, suppose $g \in L_1(\mu, \mathbf{C})$, and that g annihilates the span of F. But $g(f) = 0$ for all f in F implies $\operatorname{Re} g(f) = 0$ for all f so

$g \in F^{\perp}$ and g is purely real as a function in $L_1(\mu, \mathbf{C})$. Similarly $g(if) = 0$ for all f in F implies (ig) is purely real. Hence $g = 0$ and the span of F is dense in E. Finally to see that F is real determined, let $M = \{i\psi_{\{a\}}/\mu(\{a\})$: where a is an atom$\}$. Then each member of ext $S(L_1(\mu, \mathbf{C}))$ is of the form sm where $s \in S(\mathbf{C})$ and $m \in M$. Also from the definition of F, mf is real for each $f \in F$ and each $m \in M$. These facts show that F is real determined. □

V. **Selfadjoint Lindenstrauss spaces.** In this section we will characterize complex Lindenstrauss spaces which are linearly isometric with selfadjoint subspaces of a $C(X, \mathbf{C})$ space.

5.1. LEMMA. *Let E be a selfadjoint linear subspace of $C(X, \mathbf{C})$. If* $(\operatorname{Re} E)^* = L_1(\mu, \mathbf{R})$, *then* $E^* = L_1(\mu, \mathbf{C})$, *and* $L_1(\mu, \mathbf{R})$ *is weak*-closed in* $L_1(\mu, \mathbf{C})$.

PROOF. We can easily define a one-to-one linear mapping T of $L_1(\mu, \mathbf{C})$ onto E^* by defining, for $f \in L_1(\mu, \mathbf{R})$ and $g \in E$,

$$\hat{T}f(g) = f(\operatorname{Re} g) + if(\operatorname{Im} g). \tag{5.1}$$

Now for $h \in L_1(\mu, \mathbf{C})$ put

$$Th = \hat{T}(\operatorname{Re} h) + i\hat{T}(\operatorname{Im} h). \tag{5.2}$$

One checks that Th is a bounded linear functional on E and that T is linear. If $L \in E^*$ then $\operatorname{Re} L$ restricted to $\operatorname{Re} E$ is in $(\operatorname{Re} E)^*$, and hence is represented by some $H(L)$ in $L_1(\mu, \mathbf{R})$. Then $T[H(L) - iH(iL)]$ and L are complex functionals that agree on $\operatorname{Re} E$, and hence agree everywhere. Thus T is a one-to-one, bounded, onto linear mapping. Hence $L_1(\mu, \mathbf{C})$ is isomorphic with E^*. We can induce a new norm on E by

$$|||g||| = \sup\{|(Tf)g| : f \in S(L_1(\mu, \mathbf{C}))\}. \tag{5.3}$$

Suppose $g \in \operatorname{Re} E$, $f \in L_1(\mu, \mathbf{C})$ and s is a complex number of absolute value one such that $s(Tf)(g) = |Tfg|$; then

$$|Tf(g)| = [T(sf)](g) = [\operatorname{Re}(sf)](g) + i[\operatorname{Im}(sf)](g). \tag{5.4}$$

Since this sum is a real number,

$$|Tf(g)| = [\operatorname{Re}(sf)](g) \leq \|f\|_{L_1}\|g\|_E, \tag{5.5}$$

and $|||g||| \leq \|g\|$. But since $(\operatorname{Re} E)^* = L_1(\mu, \mathbf{R})$ we also have $\|g\| \leq |||g|||$.

We have shown therefore that there is a real linear isometry of $(\operatorname{Re} E, |||\cdot|||)$ onto $(\operatorname{Re} E, \|\cdot\|)$. By Lemma 4.5 this isometry extends to a complex isometry of $(E, |||\cdot|||)$ onto $(E, \|\cdot\|)$.

This of course implies that $S(E^*, \|\cdot\|_{L_1}) = S(E^*, \|\cdot\|_{E^*})$. For suppose $\|f\|_{E^*} > \|f\|_{L_1} = 1$. Then there is a weak*-continuous linear functional on E^* separating the weak*-compact sets $S(E^*, \|\ \|_{E^*})$ and $\{f\}$. That is there is a weak*-continuous functional L such that

$$[\operatorname{Re} L](f) > \max\{\operatorname{Re} L(g)\colon g \in S(E^*, \|\cdot\|_{E^*})\}. \tag{5.6}$$

Since L is weak*-continuous we may assume it is in E. But the right side of the inequality is equal to $\|\operatorname{Re} L\| = \|L\|$ while the left is less than

$$\sup\{[\operatorname{Re} L]h\colon \|h\|_{L_1} = 1\}$$

which is also equal to $\|L\|$, since $(E, |||\cdot|||)$ is isometric with $(E, \|\cdot\|)$. We have shown that E^* is linearly isometric to $L_1(\mu, \mathbf{C})$, and by observing their values on E we see that $L_1(\mu, \mathbf{R})$ is weak*-closed in $L_1(\mu, \mathbf{C})$. □

5.2. COROLLARY. *If $L_1(\mu, \mathbf{R})$ is a dual space $L_1(\mu, \mathbf{C})$ is a complex dual space.*

PROOF. Suppose $E^* = L_1(\mu, \mathbf{R})$. By letting $X =$ the weak*-compact set $S(E^*)$ we can embed E in $C(X, \mathbf{R})$. Then by the lemma $L_1(\mu, \mathbf{C})$ is isometric to the dual of the complex span of E in $C(X, \mathbf{C})$. □

5.3. THEOREM. *Let E be a complex L_1-predual. The following are equivalent.*

(1) *There is a compact Hausdorff space X such that E is linearly isometric to a selfadjoint subspace of $C(X, \mathbf{C})$;*

(2) *E^* is linearly isometric to an L_1-space whose real functions form a w^*-closed set;*

(3) *E has a real section.*

PROOF. All the parts of this theorem have been established in previous lemmas: (1) ⇒ (2) is Proposition 3.4 and Lemma 5.1 and, (2) ⇒ (3) is Lemma 4.10, and (3) ⇒ (1) is Lemma 4.7. □

5.4. PROPOSITION. *Let $E^* = L_1(X, \Sigma, \mu, \mathbf{C})$. If V is a real section of E then $V^* = L_1(X, \Sigma, \mu, \mathbf{R})$.*

PROOF. Suppose $V^* = L_1(Y, \theta, \nu, \mathbf{R})$. From Lemmas 4.7 and 5.1, $L_1(X, \Sigma, \mu, \mathbf{C})$ and $L_1(Y, \theta, \nu, \mathbf{C})$ are isometric. By the following lemma $L_1(X, \Sigma, \mu, \mathbf{R})$ and $L_1(Y, \theta, \nu, \mathbf{R})$ are also isometric. □

5.5. LEMMA. *If $L_1(X, \Sigma, \mu, \mathbf{C})$ is isometric with $L_1(Y, \theta, \nu, \mathbf{C})$ then $L_1(X, \Sigma, \mu, \mathbf{R})$ is isometric with $L_1(Y, \theta, \nu, \mathbf{R})$.*

PROOF. Let T be an isometry of $L_1(\mu, \mathbf{C})$ onto $L_1(\nu, \mathbf{C})$. Let ψ_A represent the characteristic function of a set A. Since T is an isometry $T(\psi_A)$ and $T(\psi_B)$ have disjoint supports whenever A and B are disjoint sets in E of finite measure. If this were not true there would be complex numbers s and t of absolute value one, a set U of positive finite measure, and an $\varepsilon > 0$ such that

$$\operatorname{Re} T(s\psi_A) > \varepsilon \quad \text{and} \quad \operatorname{Re} T(t\psi_B) > \varepsilon. \tag{5.7}$$

Hence,

$$\|T(s\psi_A) - T(t\psi_B)\| < \|T(s\psi_A)\| + \|T(t\psi_B)\| \tag{5.8}$$
$$= \|s\psi_A\| + \|t\psi_B\| = \|s\psi_A - t\psi_B\|.$$

Now for each A of finite μ measure in Σ, define

$$P(\psi_A) = |T(\psi_A)|. \tag{5.9}$$

Then P can be extended, in a complex linear fashion, to be an isometry defined on all the step functions in $L_1(\mu, \mathbf{C})$, and therefore, of $L_1(\mu, \mathbf{C})$ into $L_1(\nu, \mathbf{C})$. Also P carries $L_1(\mu, \mathbf{R})$ into $L_1(\nu, \mathbf{R})$. To see that P carries $L_1(\mu, \mathbf{R})$ onto $L_1(\nu, \mathbf{R})$, let f be a nonnegative function in $L_1(\nu, \mathbf{R})$. Since T is an isometry there are scalars a_j and disjoint sets $A_j \in \Sigma$ of μ-finite measure such that

$$\int \left| \sum a_j T(\psi_{A_j}) - f \right| d\nu < \varepsilon. \tag{5.10}$$

It follows that

$$\int \left| \operatorname{Im} \sum a_j T(\psi_{A_j}) \right| d\nu < \varepsilon, \tag{5.11}$$

and

$$\int \left| \operatorname{Re} \sum a_j T(\psi_{A_j}) \wedge 0 \right| d\nu < \varepsilon. \tag{5.12}$$

We therefore have that

$$3\varepsilon > \int \left| \left| \sum a_j T(\psi_{A_j}) \right| - f \right| d\nu = \int \left| \sum |a_j| |T(\psi_{A_j})| - f \right| d\nu \tag{5.13}$$
$$= \int \left| \sum |a_j| P(\psi_{A_j}) - f \right| d\nu = \int \left| P\left(\sum |a_j| \psi_{A_j} \right) - f \right| d\nu. \quad \square$$

5.6. Corollary. *Real sections of a complex Lindenstrauss space have isometric duals.*

Definition. Let u be an extreme point of the unit ball of a Banach space E. The *state space* associated with u is

$$K(u) = \{ f \in S(E^*) : f(y) = 1 \}. \tag{5.14}$$

5.7. Corollary. *Let* $E^* = L_1(\mu, \mathbf{C})$, $u \in \operatorname{ext} S(E)$, *and* $G =$ *the closed real linear span of* $K(u)$. *Then* G *is isometric with* $L_1(\mu, \mathbf{R})$.

Proof. From the work of Hirsberg and Lazar [Theorems 1.1 and 1.2] we may assume that $E \subseteq C(X, \mathbf{C})$, $u = 1 \in E$, and E is selfadjoint. So by Propositions 3.4 and 5.4 we want to show that G is isometric with $(\operatorname{Re} E)^*$. The natural candidate for an isometry from G to $(\operatorname{Re} E)^*$ is the restriction mapping T which carries a functional in G to its restriction to $\operatorname{Re} E$. Clearly T is a real linear mapping. It is one-to-one since the complex space of $\operatorname{Re} E$ is all of E.

To see that T is onto, let $f \in (\operatorname{Re} E)^*$. Extend f to be a linear functional on $C(X, \mathbf{R})$ with the same norm. Let f^+ and f^- be the positive and negative parts of the extension of f. Let h^+ and h^- be the restrictions of f^+ and f^- to E. Then h^+ and h^- are in G,

$$h = h^+ - h^- \in G, \tag{5.15}$$

and $Th = f$.

For $g \in G$, $\|g\| \geqslant \|Tg\|$. It remains to show that $\|Tg\| \geqslant \|g\|$. Let $f \in (\operatorname{Re} E)^*$. Let f^+, f^-, h^+, h^- and h be as defined in the last paragraph. We want to show that $\|h\| \leqslant \|f\|$. Let ε be larger than zero. Let $k \in S(E)$ be such that $|h(k)| \geqslant \|h\| - \varepsilon$. In fact by multiplying k by an appropriate constant we may also assume that $h(k) \geqslant 0$. But

$$h(k) = h(\operatorname{Re} k) + i\big[f^+(\operatorname{Im} k) - f^-(\operatorname{Im} k)\big]. \tag{5.16}$$

Since f^+ and f^- are real functionals and $h(k)$ is real, the second term is zero. Since $\operatorname{Re} k \in \operatorname{Re} E$,

$$f(\operatorname{Re} k) = h(\operatorname{Re} k) \geqslant \|h\| - \varepsilon. \tag{5.17}$$

Hence $\|f\| \geqslant \|h\|$. $\square$

VI. **A complex Lindenstrauss space without real sections.** In this section we construct an example of a complex Lindenstrauss space that is not isometric with a selfadjoint subspace of a $C(X, \mathbf{C})$ space. The space itself is easy to describe. The verification uses most of the previous results on real sections, and the Borsuk antipodal mapping theorem.

6.1. Example. Let

$$K = S^2 = \{(\rho e^{i\theta}, \beta): \rho^2 + \beta^2 = 1, \text{and } 0 \leqslant \theta < 2\pi\}. \tag{6.1}$$

For

$$k = (\rho e^{i\theta}, \beta) \in K, \tag{6.2}$$

put

$$s(k) = (\rho e^{i[\theta + \pi/2]}, -\beta), \tag{6.3}$$

and let

$$E = \{f \in C(K, \mathbf{C}): f(s(k)) = if(k) \text{ for all } k \in K\}. \tag{6.4}$$

We will show that E is a complex Lindenstrauss space which is not isometric to a selfadjoint subspace of some $C(X, \mathbf{C})$.

Claim. (1) E is a complex Lindenstrauss space,

(2) $\operatorname{ext}(S(E^*))$ is weak*-compact,

(3) if $y \notin \theta[k] = \{s^n(k): n = 0, 1, 2, 3\}$ then there is a neighborhood U of y, and an $f \in E$ such that $f(k) = 1$ and f vanishes on U, and

(4) if, when viewed as functionals on E, $\delta(y) \not\in \{\alpha\delta(k): \alpha \in \mathbf{C}_j, |\alpha| = 1\} \subseteq E^*$, then $y \not\in \theta[k]$.

PROOF. (1) Let

$$Pf(k) = \left[\sum_{h=0}^{3} (-i)^n f(s^n(k))\right] \Big/ 4. \tag{6.5}$$

Then P is a norm one projection of $C(K, \mathbf{C})$ onto E. For $x \in C(K, \mathbf{C})^*$ let $V(x)$ be the restriction of x to E. Then $P^* \circ V$ is a norm one projection of $C(K, \mathbf{C})^*$ onto an isometric copy of E^*. Hence E^* is also an L-space [see, for example, Lacey, 1974, Chapter 6, §17].

(2) This follows by verifying that x is an extreme point of $s(E^*)$ if and only if there is an $s \in \mathbf{C}$ of absolute value one, and a $k \in K$ such that $x = s\delta(k)$.

(3) Let U and V be a neighborhood of y and k such that the eight sets $\{s^n(U), s^n(V)\}_{n=0}^{3}$ are disjoint. Let f be a continuous function that is 1 at k and zero off V. Let

$$g = \begin{cases} (-i)^n f & \text{on } s^n(V), \\ 0 & \text{elsewhere.} \end{cases} \tag{6.6}$$

Then g is the desired function.

(4) This follows directly from the definitions of E and s, and was singled out for a future reference. □

Now suppose T is an isometric isomorphism of F onto E where F is a selfadjoint subspace of $C(X, \mathbf{C})$ for some compact Hausdorff space X. Let

$$F^* = L_1(X, \Sigma, \mu, \mathbf{C}). \tag{6.7}$$

From Theorem 5.3, $L_1(X, \Sigma, \mu, \mathbf{R})$ is a weak*-closed subset.

For $x \in K$, $T^*\delta(x) \in \operatorname{ext} S(F^*)$. Hence there is a complex number $q(x)$ and an atom $a(x) \in (X, \Sigma, \mu)$ such that

$$T^*\delta(x) = q(x)b(x) \tag{6.8}$$

where

$$b(x) = \psi_{\{a(x)\}}/\mu(a(x)) \tag{6.9}$$

and

$$|q(x)| = 1. \tag{6.10}$$

Now for z and w in $\mathbf{C}$ define

$$z \approx w \quad \text{if and only if } z = -w. \tag{6.11}$$

If S^1 is the unit circle, then, by the natural mapping Q, $S^1/\approx$ is equivalent to $[0, \pi]$ mod π.

CLAIM. (5) $Q \circ q(\cdot)$ is a continuous mapping of K onto $[0, \pi]$ mod π.

(6) $Q \circ q(sk) = [\pi/2 + Q \circ q(b)]$ mod π.

PROOF. (5) Suppose $x_j \to x$. By compactness we may choose a subsequence, which we assume we already have such that $b(x_j)$ converges to something, say b, and $q(x_j)$ converges to, say q. Now

$$b(x_j) \in \operatorname{ext} S(F^*) \cap L_1(X, \Sigma, \mu, \mathbf{R}) \tag{6.12}$$

which is weak*-compact by Claim (2) and Theorem 5.3. Hence there is an atom, a, such that either

$$b = \psi_{\{a\}}/\mu(a) \quad \text{or} \quad b = -\psi_{\{a\}}/\mu(a). \tag{6.13}$$

But if $a \neq a(x)$, then

$$\varnothing = \{\alpha b\colon \alpha \in \mathbf{C}, |\alpha| = 1\} \cap \{\alpha b(x)\colon \alpha \in \mathbf{C}, |\alpha| = 1\}. \tag{6.14}$$

Hence by Claims (4) and (3) there is an $f \in F$ which vanishes on $\{\alpha b\colon \alpha \in \mathbf{C}, |\alpha| = 1\}$, and for which $[b(x)]f = 1$. This is not possible though since $|q| = 1 = |q(x)|$, and

$$qb = \lim_{j\to\infty} q(x_j)b(x_j) = \lim T^*(\delta(x_j)) = T^*(\delta(x)) = q(x)b(x), \tag{6.15}$$

where the limits are taken in the weak*-topology. We conclude that $a = a(x)$; and $b = b(x)$ or $b = -b(x)$. It then follows from (6.15) that $q = q(x)$ or $q = -q(x)$. In either case we have that

$$\lim_{j\to\infty} Q \circ q(x_j) = Q \circ q(x). \tag{6.16}$$

This establishes the first assertion.

(6) This claim follows from the definitions of q, b, and Q and from the corresponding fact for T^*, that for $f \in F$, $Tf \in E$ so

$$T^*\delta(sk)f = (Tf)(sk) = i(Tf)(k) = iT^*\delta(k)(f). \quad \square \tag{6.17}$$

Now for $k = (\rho e^{i\theta}, \beta) \in K$, put

$$m(k) = [2Q \circ q(\rho e^{i\theta/2}, \beta)] \bmod 2\pi. \tag{6.18}$$

The above shows that m is a continuous mapping of S^2 into S^1. Furthermore if $k = (\rho e^{i\theta}, \beta) \in K$,

$$\begin{aligned} m(-k) &= m(\rho e^{i[\theta+\pi]}, -\beta) = 2Q \circ q(\rho e^{i[\theta+\pi]/2}, -\beta) \\ &= 2Q \circ q(s(\rho e^{i\theta/2}, \beta)) = [\pi + m(k)] \bmod 2\pi \\ &= -m(k). \end{aligned} \tag{6.19}$$

Hence m is an antipodal mapping from S^2 to S^1. But Borsuk's antipodal mapping theorem no such mapping exists [Dugundji, 1966, p. 349, Corollary 6.2].

Hence E is not isometric to a selfadjoint subspace of a $C(X, \mathbf{C})$ space. $\square$

A modification of the last example is also of some interest. As a corollary to their work (Theorems 1.1 and 1.2) Hirsberg and Lazar observed that analogous to the real case, the following is true.

6.2. PROPOSITION. *A complex Banach space E is isometric to a $C(X, \mathbf{C})$ space if and only if*

(i) *E is a Lindenstrauss space,*

(ii) $\operatorname{ext}(s(E^*))$ *is w^*-closed, and*

(iii) $\operatorname{ext} S(E) \neq \varnothing$.

Clearly solution (i) cannot be simply dropped from the list. Proposition 3.4 and examples from the real case show that (ii) cannot be dropped. The following shows that, as expected, also (iii) cannot be dropped.

6.3 EXAMPLE. Let $X = S^2$ and

$$E = \{f \in C(X, \mathbf{C}): f(x) = -f(-x)\}. \tag{6.20}$$

CLAIM. (1) E is a selfadjoint Lindenstrauss space contained in $C(X, \mathbf{C})$;

(2) $\operatorname{ext} S(E^*)$ is weak*-compact;

(3) $S(E)$ does not contain an extreme point.

PROOF. It is clear that E is selfadjoint. The rest of (1) and (2) is established as in the last example.

(3) Suppose $u \in \operatorname{ext} S(E)$. By a result of Hirsberg and Lazar $x(u) = 1$ for all $x \in \operatorname{ext} S(E^*)$. In particular $m(k) = u(k)$ is a continuous antipodal mapping of S^2 into S^1 and again by Borsuk's theorem no such mapping is possible. $\square$

REFERENCES

1. E.M. Alfsen and E.G. Effros, *Structure in real Banach spaces*. I, II, Ann. of Math (2) **96** (1972), 98–173. MR **50** #5432.

2. Y. Benyamini and J. Lindenstrauss, *A predual of l_1 which is not isomorphic to a $C(K)$ space*, Israel J. Math. **13** (1973), 246–254. MR **48** #9348.

3. E. Bishop, *A generalization of the Stone-Weierstrass theorem*, Pacific J. Math. **11** (1961), 777–783. MR **24** #A3502.

4. J. Dieudonné, *Sur les espaces l_1*, Arch. Math. **10** (1959), 151–152. MR **21** #3763.

5. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR **33** #1824.

6. E. G. Effros, *Structure in simplexes*, Acta Math. **117** (1967), 103–121. MR **34** #3287.

7. ______, *On a class of complex Banach spaces*, Illinois J. Math. **18** (1974), 48–59. MR **48** #6890.

8. H. Fakhoury, *Préduaux de L-espace: Notion de centre*, J. Functional Analysis **9** (1972), 189–207. MR **46** #2398.

9. A. Grothendieck, *Une caractérisation vectorielle-métrique des espaces L^1*, Canad. J. Math. **7** (1955), 552–561. MR **17**, 877.

10. B. Hirsberg, and A.J. Lazar, *Complex Lindenstrauss spaces with extreme points*, Trans. Amer. Math. Soc. **186** (1973), 141–150. MR **48** #11996.

11. S. Kakutani, *Concrete representation of abstract (M)-spaces*, Ann. of Math. (2) **42** (1941), 994–1024. MR **3**, 205.

12. ______, *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Ann of Math. (2) **42** (1941), 523–537. MR **2**, 318.

13. E. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, New York, 1974.

14. A.J. Lazar and J. Lindenstrauss, *On Banach spaces whose duals are L_1 spaces*, Israel J. Math. **4** (1966), 205–207. MR **34** #6488.

15. A.J. Lazar, *The unit ball in conjugate L_1 spaces*, Duke Math. J. **39** (1972), 1–8. MR **46** #2380.

16. A.J. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1 spaces and their representing matrices*, Acta Math. **126** (1971), 165–193. MR **45** #862.

17. D.R. Lewis, and C. Stegall, *Banach spaces whose duals are isomorphic to $l_1(\Gamma)$*, J. Functional Anal. **12** (1973), 177–187. MR **49** #7731.

18. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Lecture Notes in Math., Vol. 338, Springer-Verlag, Berlin and New York, 1973.

19. J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. No. 48, 1964. MR **31** #3828.

20. J. Lindenstrauss and D. E. Wulbert, *On the classification of the Banach spaces whose duals are L_1 spaces*, J. Functional Analysis **4** (1969), 332–349. MR **40** #3274.

21. Å. Lima, *Complex Banach spaces whose duals are L_1-spaces*, Israel J. Math. **24** (1976), 59–72.

22. E. Michael and A. Pełczyński, *Peaked partition subspaces of $C(X)$*, Illinois J. Math **11** (1967), 555–562. MR **36** #670.

23. E. A. Michael and A. Pełczyński, *Separable Banach spaces which admit l_n^∞ approximations*, Israel J. Math. **4** (1966), 189–198. MR **35** #2129.

24. N.J. Nielsen and G.H. Olsen, *Complex preduals of L_1 and subspaces of $l_\infty^n(C)$*, Mat. Inst. Univ. Oslo (preprint).

25. G.H. Olsen, *On the classification of complex Lindenstrauss spaces*, Mat. Inst. Univ. Oslo **16** (1973).

26. R. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N. J., 1966. MR **33** #1690.

27. C. Samuel, *Sur certains espaces $C_\sigma(S)$ et sur les sous-espaces complementes de $C(S)$*, Bull. Sci. Math. (2) **95** (1971), 65–82.

28. C. Stegall, *Banach spaces whose duals contain $l_1(\Gamma)$ with applications to the study of dual $L_1(\mu)$ spaces*, Trans. Amer. Math. Soc. **176** (1973), 463–477. MR **47** #3953.

29. D.E. Wulbert, *Convergence of operators and Korovkin's theorem*, J. Approximation Theory **1** (1968), 381–390. MR **38** #3679.

30. M. Zippin, *On some subspaces of Banach spaces whose duals are L_1 spaces*, Proc. Amer. Math. Soc. **23** (1969), 378–385. MR **39** #7400.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CALIFORNIA 92093